AD-A187 601 A QUEUEING SYSTEM WITH INDEPENDENT MARKOV INPUT STREAMS 1/1
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ENGINEERING I STAVRAKAKIS ET AL 1987
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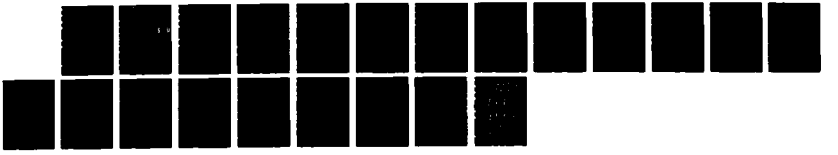

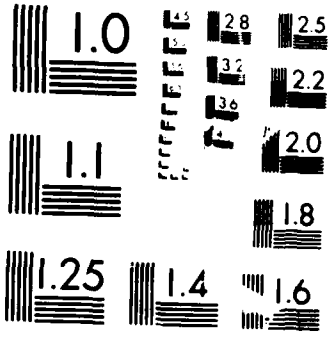

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A Technical Report
Contract No. AFOSR 87-0095
January 1, 1987 - December 31, 1987

A QUEUEING SYSTEM WITH INDEPENDENT MARKOV INPUT STREAMS

AFOSR-TR- 87-1619

Submitted to:

Air Force Office of Scientific Research, NM
Building 410
Bolling Air Force Base
Washington, D.C. 20332-6448

Attention: Major Brian W. Woodruff
USAF, NM

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Report No. UVA/525677/EE87/101

April 1987

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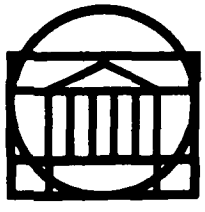

SCHOOL OF ENGINEERING AND
APPLIED SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA 22901

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS: NONE
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release, distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): UVA/525677/EE87/101	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 87-1619
6a. NAME OF PERFORMING ORGANIZATION: University of Virginia Department of Electrical Engr.	6b. OFFICE SYMBOL (If applicable):
6c. ADDRESS (City, State, and ZIP Code): Department of Electrical Engineering Thornton Hall Charlottesville, VA 22901	7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research, PKZ
	7b. ADDRESS (City, State, and ZIP Code): Building 410 Bolling Air Force Base Washington, D.C. 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Air Force Office of Scientific Research	8b. OFFICE SYMBOL (If applicable): NM
	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-87-0095
8c. ADDRESS (City, State, and ZIP Code): Building 410 Bolling Air Force Base Washington, D.C. 20332-6448	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

A Queueing System with Independent Markov Input Streams

12. PERSONAL AUTHOR(S)
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13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM 01/01/87 TO 12/31/87	1987 April 24	16

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In this work, a discrete time single server queueing system with arbitrary (finite) number of input streams, is considered. The input streams are assumed to be independent but successive arrivals in a single stream are not. More specifically, it is assumed that for each stream, arrivals are governed by an underlying finite state space Markov chain and that a visit to a state corresponds to one or none arrivals, according to a stationary mapping rule. The first in-first out service policy is adopted.

For the system described above we develop a method to calculate the average number of customers in the queueing system. Then, the mean time that a customer spends in the system is calculated by using Little's theorem.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Major Brian W. Woodruff	22b. TELEPHONE (Include Area Code): (202) 767-5025
	22c. OFFICE SYMBOL: NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

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Copy No. 11

April 1987

I. Introduction

We consider a discrete time single server queueing system that is fed by N independent input streams (Fig.1). The service time, T, is constant and equal to one, which is the distance between successive arrival points. The first in - first out policy is adopted and the buffer size is infinite.

When successive arrivals in each input stream are independent, the above queueing system has been studied, [1], and the mean time, D_I, that a customer spends in the system was then found to be given by the following expression.

$$D_I = \left[1 + \frac{\sum_{n=1}^{N} \sum_{m>n}^{N} \alpha_n \alpha_m}{(1 - \sum_{n=1}^{N} \alpha_n) \cdot \sum_{n=1}^{N} \alpha_n} \right] \qquad (*)$$

where α_n is the probability of an arrival at the input of the queue, and corresponds to the mean arrival rate of the Bernoulli process that describes the arrival streams.

If the arrival process is a first order ergodic Markov chain with state space S = {0, 1}, where 1 corresponds to an arrival and 0 to the absence of such an event, then the average time, D_M, that a customer spends in the system is given by the expression below, [2], [3].

$$D_M = \left[1 + \frac{\sum_{n=1}^{N} \sum_{m>n}^{N} \alpha_n \alpha_m \left(1 + \frac{\gamma_n}{1-\gamma_n} + \frac{\gamma_m}{1-\gamma_m} \right)}{(1 - \sum_{n=1}^{N} \alpha_n) \cdot \sum_{n=1}^{N} \alpha_n} \right] \qquad (**)$$

where $\gamma_n = P(1/1) - P(1/0)$ and where α_n is the arrival rate given by the expression

$$\alpha_n = \frac{P(1/0)}{1-\gamma_n}.$$

P(/) denotes conditional probability.

In this paper, it is assume that each input stream is described by a finite-state Markov chain. The cardinality of the state space of the Markov chain associated with the i^{th} stream is denoted by M_i. The arrival process of each input stream corresponds then to a mapping from a finite-state Markov chain onto the set {0,1}, where 1 represents a single arrival and where 0 represents no arrival. In this case, the average time that a customer spends in the system is obtained from the solution of $M_1 x M_2 x \cdots x M_N$ linear equations.

Clearly, the Markov arrival system described in [2] is a special case of the general system considered in this paper. In [2], the underlying Markov chain has two states only, and actually coincides with the arrival process. The closed form solution obtained in [2], for the average time that a customer spends in the system, does not extend to the case where multi-state Markov chains are present. The two state Markov model gives rise to a second order equation, whose roots are used in the derivation of the closed form solution. This procedure does not extend to a larger state space Markov model, since then expressions for the roots of high order equations would be needed.

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The queueing system with Bernoulli arrivals, is also a special case of the system considered in this paper. In that case, the arrival process coincides with the underlying Markov process and the state transition probabilities are properly selected. Then, D_l can be derived from the solution of 4 linear equations.

The system considered here has several applications. For example, the single server may correspond to some central node which accepts and processes packets originating from several random-access communication systems. Each input stream represents then

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the output process from a random-access communication system; that is, the process of the successfully transmitted within the latter system packets. Then, the per stream Bernoulli model is unrealistic, and the two-state Markov model can be inefficient, [4]. On the other hand, a three-state Markov model per input stream may represent an efficient approximation and may be intuitively pleasing. For example, in synchronous packet random-access systems, the three states may correspond to the idle versus success versus collision channel states per slot. Each input stream is then governed by this three-state process, whose characteristics are induced by the deployed random-access algorithm. The above three-state process is not necessarily Markov, but can be closely approximated by a Markov such process, especially in the presence of heavy traffic, [4].

II. The general queueing system

The general configuration of the problem that will be described in this section appears in Fig 1. The system consists of N input streams which feed a single server. The server has an infinite capacity buffer.

The arrival processes $\{a_i^j\}_{j\geq 0}$, i = 1, 2, $\cdots$, N, are assumed to be synchronized discrete time processes, and at most one arrival can occur in each input line per unit time. The time separation between successive possible arrival points is constant and equal to one. The arrival processes $\{a_i^j\}_{j\geq 0}$, i = 1, 2, $\cdots$, N, may represent the output processes of multi-user random access slotted communication networks, where the arrival points coincide then with the ends of slots. It is obvious that in the latter system, the condition of having at most one packet arrival per input stream and per unit time is satisfied.

The first in - first out (FIFO) policy is adopted and the service time is assumed to be constant and equal to the distance separation of successive arrival points. More than one arrivals (from different input streams) that occur at the same arrival point are served in a randomly chosen order. In the packet communication system, the service time policy implies that arriving and departing packets have the same length.

Let $\{x_i^j\}_{j\geq 0}$ denote a discrete time ergodic Markov process associated with the i^{th} input stream, with finite state space $S^i = \{\sigma_i^1, \cdots \sigma_i^{M_i}\}$. Let also a_i be a stationary mapping rule from the set S_i onto the set $\{0, 1\}$, where 1 corresponds to an arrival and 0 to the absence of such an event. Then the arrival process of the i^{th} input stream is

$$\{a_i^j\}_{j\geq 0} = \{a_i(x_i^j)\}_{j\geq 0} = \{a_i(x_i^0), a_i(x_i^1), \cdots\}.$$

From the description of the arrival process it is implied that successive arrivals from the same input stream are not independent, but they are governed by an underlying finite state Markov chain, $\{x_i^j\}_{j\geq 0}$, and a stationary mapping rule a_i.

In this system, it is assumed that the processes $\{x_i^j\}_{j\geq 0}$, $i = 1, 2, \cdots, N$, are mutually independent and thus the arrival processes $\{a_i^j\}_{j\geq 0}$, $i = 1, 2, \cdots, N$, are also independent. If $\{b^j\}_{j\geq 0} = \{b^0, b^1, \cdots\}$ is the process that describes the total arrivals occuring at a single arrival point, then

$$b^j = \sum_{i=1}^{N} a_i(x_i^j) \quad , \quad j = 0, 1, 2, \cdots \tag{2}$$

and $b^j \in \{0, 1, 2, \cdots, N\}$.

Referring to the example of the output process of a multi-user random access communication network, we may define $\{\tilde{x}_i^j\}_{j\geq 0}$ to be the process that describes the state of the channel at the end of a slot. Let us consider a ternary channel state space

$$S^i \equiv \{0, 1, 2\}$$

where, 0, 1 and 2, respectively, denote that 0, 1, or more than 1 packets attempted packet transmission in a single slot. Since a packet appears in the output process only if it is the only one transmitted within the corresponding slot, the arrival process $\{a_i^j\}_{j\geq 0}$ can be clearly described via the mapping

$$a_i(\tilde{x}_i^j) = \begin{cases} 1 & \text{if} \quad \tilde{x}_i^j = 1 \\ 0 & \text{if} \quad \tilde{x}_i^j = 0, 2 \end{cases}$$

The process $\{\tilde{x}_i^j\}_{j\geq 0}$ is controlled by the deployed random-access algorithm, and is generally non-Markov. However, this process can be approximated by a Markov process $\{x_i^j\}_{j\geq 0}$ which has the same state space as $\{\tilde{x}_i^j\}_{j\geq 0}$ and is ergodic within the stability region of the random-access algorithm. Simulation results, [4], have shown that, for at least a class of random-access algorithms, the three-state Markov model provides a good approximation of the process $\{\tilde{x}_i^j\}_{j\geq 0}$.

III. Analysis of the queueing system

This section is devoted to the analysis of the system described in the previous section. The main result of the analysis is the derivation of the linear equations whose solution give the mean number of packets in the system. This result, in conjunction with Little's formula, provide the mean time that a packet spends in the system.

Let $\pi_i(k)$ and $p_i(k, j)$, $k,j \in S^i$, denote the steady state and the transition probabilities of the ergodic Markov chain, $\{x_i^j\}_{j\geq 0}$, $i = 1, 2, \cdots, N$. Let also $p^n(j\,;\bar{y})$ denote the joint probability that there are j packets in the system at the n^{th} arrival point (arrivals at that point are included) and the states of the Markov chains are $y_1, y_2, \cdots, y_N$, where $\bar{y} = (y_1, y_2, \cdots, y_N)$. The vector $\bar{y}$ describes the state of a new ergodic Markov chain that

is generated by the N independent Markov chains described before, with steady state and transition probabilities $\pi(\bar{y})$ and $p(\bar{x}, \bar{y})$ respectively, and with state space $\bar{S} = S^1 x S^2 x \cdots x S^N$.

The operation of the system can be descibed by an N + 1 dimensional (infinite state space) Markov chain imbedded at the arrival points, with state space T = (0, 1, 2, $\cdots$)x $\bar{S}$ and state probabilities given by the following recursive equations

$$p^n(j ; \bar{y}) = \sum_{\bar{x} \varepsilon \bar{S}} p^{n-1} \left(j+1 - \sum_{i=1}^{N} a_i(x_i) ; \bar{x}\right) p(\bar{x}, \bar{y}) \quad , \quad j \geq N+1$$

or (3)

$$p^n(j ; \bar{y}) = \sum_{k=1}^{j+1} \sum_{\bar{x} \varepsilon F_{j+1-k}} p^{n-1}(k ; \bar{x})\, p(\bar{x}, \bar{y}) + \sum_{\bar{x} \varepsilon F_j} p^{n-1}(0 ; \bar{x})\, p(\bar{x}, \bar{y}) \quad , \quad 0 \leq j \leq N$$

where

$$F_v = \{\bar{x} = (x_1, x_2, \cdots x_N) \ \varepsilon \ \bar{S} : \sum_{i=1}^{N} a_i(x_i) = v\} \tag{4}$$

There are totally $M_1 x M_2 x \cdots x M_N$ equations given by (3) for a fixed j and all $\bar{y} \varepsilon \bar{S}$, where M_i is the cardinality of S^i, i = 1, 2, $\cdots$, N.

The original assumption concerning the ergodicity of the Markov chains associated with the input streams implies the ergodicity of the arrival processes $\{a_i^j\}_{j \geq 0}$, i = 1, 2, $\cdots$, N. The latter together with the well known condition, [5],

$$\sum_{i=1}^{N} \sum_{x_i \varepsilon S^i} \pi_i(x_i : a_i(x_i) = 1) < 1 \tag{5}$$

imply that the Markov chain described in (3) is ergodic and there exist steady state (equilibrium) probabilities. Thus, we can consider the limit of the equations in (3) as n approaches infinity and obtain similar equations for the steady state probabilities.

By considering the steady state probabilities given by (3) and manipulating the resulting equations, as it is shown in Appendix A, we obtain the following system of linear equations

$$P(z\,;\,\bar{y}) = \sum_{v=0}^{N} \sum_{\bar{x}\,\varepsilon\,F_v} z^{v-1}\,[P(z\,;\,\bar{x}) + (z-1)\,p\,(0\,;\,\bar{x})]\,p(\bar{x},\bar{y}) \quad , \quad \bar{y}\,\varepsilon\,\bar{S} \tag{6}$$

where $P(z\,;\,\bar{y})$ is the generating function of the steady state distribution of the N+1 dimensional imbedded Markov chain, given in Appendix A.

From the independence of the Markov chains associated with the input streams and the state description of the imbedded N+1 dimensional Markov chain, it is obvious that

$$\pi(\bar{x}) = \prod_{i=1}^{N} \pi_i(x_i) \quad , \quad p(\bar{x},\bar{y}) = \prod_{i=1}^{N} p_i(x_i, y_i) \tag{7}$$

$$p(0\,;\,\bar{x}) = p_0 \prod_{i=1}^{N} \pi_i(x_i) \tag{7b}$$

If P(z) is the generating function of the distribution of the number of packets in the system, then

$$P(z) = \sum_{j=0}^{\infty} p(j)z^j = \sum_{j=0}^{\infty} \sum_{\bar{y}\,\varepsilon\,\bar{S}} p(j\,;\,\bar{y})z^j = \sum_{\bar{y}\,\varepsilon\,\bar{S}} P(z\,;\,\bar{y}) \tag{8a}$$

and

$$P'(z) = \sum_{\bar{y}\,\varepsilon\,\bar{S}} P'(z\,;\,\bar{y}) \tag{8b}$$

where $P'(z\,;\,y)$, $\bar{y}\,\varepsilon\,\bar{S}$, can be derived by differentiating (6) and are given by

$$P'(z\,;\,\bar{y}) = \sum_{v=0}^{N} \sum_{\bar{x}\,\varepsilon\,F_v} \Big[(v-1)z^{v-2}\,[P(z\,;\,\bar{x}) + (z-1)p(0\,;\,\bar{x})] + z^{v-1}\,[P'(z\,;\,\bar{x}) + p(0\,;\,\bar{x})] \Big]\,p(\bar{x},\bar{y}) \tag{9}$$

for $\bar{y}\,\varepsilon\,\bar{S}$.

Since $P'(1)$ is the average number of packets in the system, Q, from (8b) we have that

$$Q = \sum_{\bar{y} \varepsilon \bar{S}} P'(1 ; \bar{y}) \quad . \tag{10}$$

$P'(1 ; \bar{y})$, $\bar{y} \varepsilon \bar{S}$, are in fact the solutions of the following $M_1 x \cdots x M_N$ dimensional linear system of equations, which are obtained from (9) by setting z=1 :

$$P'(1 ; \bar{y}) = \sum_{v=0}^{N} \sum_{\bar{x} \varepsilon F_v} \left[(v-1) P(1 ; \bar{x}) + P'(1 ; \bar{x}) + p(0 ; \bar{x}) \right] p(\bar{x} , \bar{y}) \quad , \quad \bar{y} \varepsilon \bar{S} \tag{11}$$

where

$$P(1 ; \bar{x}) = \pi(\bar{x}) = \prod_{i=1}^{N} \pi_i(x_i) \tag{12}$$

The $M_1 x \cdots M_N$ linear equations with respect to $\bar{y} \varepsilon \bar{S}$ that appear in (11) are linearly dependent. This is usually the case when the equations have been derived from the state transition description of a Markov chain. By following the procedure that is shown in Appendix B, we obtain an additional linear equation with respect to $P'(1 ; \bar{y})$, $\bar{y} \varepsilon \bar{S}$, which is linearly independent from those in (11) and is given by

$$\sum_{v=0}^{N} \sum_{\bar{x} \varepsilon F_v} \left[2(v-1) P'(1 ; \bar{x}) + (v-1)(v-2) P(1 ; \bar{x}) + 2(v-1) p(0 ; \bar{x}) \right] = 0 \tag{13}$$

In appendix B we also calculate the steady state probability that there is no packet in the system, denoted by p_0. As it was expected it was found that

$$p_0 = 1 - \sum_{i=1}^{N} \sum_{x_i \varepsilon S^i} \pi_i(x_i : a_i(x_i) = 1) \tag{14}$$

By substituting (7b), (12) and (14) into (11) and (13) and solving the $M_1 x \cdots x M_N$ dimensional linear system of equations that consists of (13) and any $M_1 x \cdots x M_N - 1$ equations taken from (11), we compute $P(1 ; \bar{x})$, $\bar{x} \varepsilon \bar{S}$. Then, the average number of packets in the system can be computed by (10).

The average time, D, that a packet spends in the system can be obtained by using Little's formula and it is given by

$$D = \frac{Q}{\sum_{i=1}^{N} \sum_{x_i \varepsilon S^i} \pi_i (x_i : a_i(x_i) = 1)} \tag{15}$$

The denominator in the above expression corresponds to the total input traffic to the queueing system.

IV. Conclusions

The main result of this paper is a method to calculate the mean time that a customer spends in the system, for a queueing system with many input streams and arrivals that depend on the state of arbitrary Markov chains associated with the input streams. That result can be obtained by solving $M_1 x \cdots x M_N$ linear equations, where M_i is the cardinality of the Markov chain associated with the i^{th} input stream.

As a possible application, it has been demonstrated how the queueing system under consideration can approximate a queueing system fed by the output processes of independent multi-user random access communication networks.

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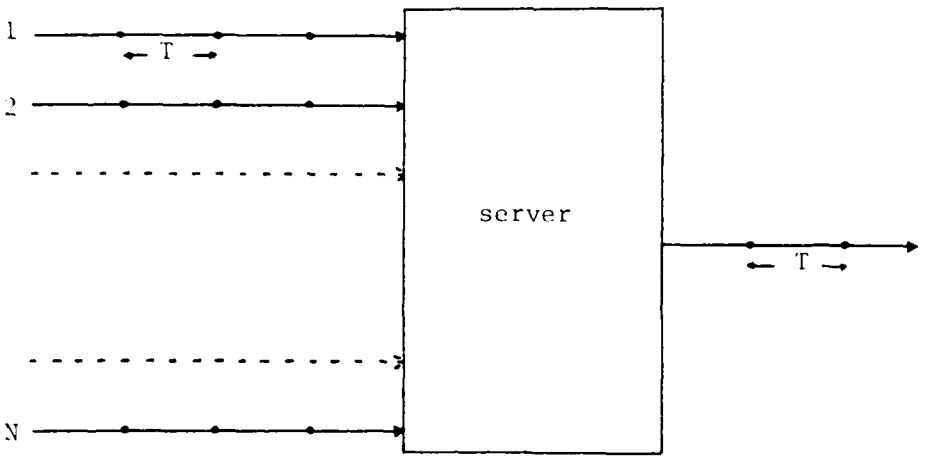

Figure 1.

Appendix A

In this section we derive the linear equations that are given by (6). We write the steady staty probabilities (under ergodicity) by considering the limit of (3) as $n \to \infty$ and obtain

$$p(j\,;\,\bar{y}) = \sum_{\bar{x}\in\bar{S}} p\left(j+1-\sum_{i=1}^{N} a_i(x_i)\;;\;\bar{x}\right) p(\bar{x},\bar{y}) \quad , \quad j \geq N+1$$

$$p(j\,;\,\bar{y}) = \sum_{k=1}^{j+1} \sum_{\bar{x}\,\varepsilon\,F_{j+1-k}} p(k\,;\,\bar{x})\, p(\bar{x},\bar{y}) + \sum_{\bar{x}\,\varepsilon\,F_j} p(0\,;\,\bar{x})\, p(\bar{x},\bar{y}) \quad , 0 \leq j \leq N$$

If $P(z\,;\,\bar{y})$ is the z-transform of the joint probability distribution that there are j packets in the system and the Markov chain is in state $\bar{y}$, defined by

$$P(z\,;\,\bar{y}) = \sum_{j=0}^{\infty} p(j\,;\,\bar{y}) z^j = \sum_{j=0}^{\infty} p(j\,;\,\bar{y}) z^j,$$

then, from (A_1) we obtain

$$P(z\,;\,\bar{y}) = \sum_{j=0}^{N} p(j\,;\,\bar{y}) z^j + \sum_{j=N+1}^{\infty} \sum_{\bar{x}\,\varepsilon\,\bar{S}} p\left(j+1-\sum_{i=1}^{N} a_i(x_i)\;;\;\bar{x}\right) p(\bar{x}\,,\,\bar{y})\, z^j$$

$$= \sum_{j=0}^{N} p(j\,;\,\bar{y}) z^j + \sum_{\nu=0}^{N} \sum_{j=N+1}^{\infty} \sum_{\bar{x}\,\varepsilon\,F_\nu} p(j+1-\nu\,;\,\bar{x})\, p(\bar{x}\,,\,\bar{y}) z^j$$

$$= \sum_{j=0}^{N} p(j\,;\,\bar{y}) z^j + \sum_{\nu=0}^{N} \sum_{k=N+2-\nu}^{\infty} \sum_{\bar{x}\,\varepsilon\,F_\nu} p(k\,;\,\bar{x})\, p(\bar{x},\bar{y}) z^k\, z^{\nu-1}$$

$$= \sum_{j=0}^{N} p(j\,;\,\bar{y}) z^j + \sum_{\nu=0}^{N} \sum_{\bar{x}\,\varepsilon\,F_\nu} z^{\nu-1} \left[\sum_{k=N+2-\nu} p(k\,;\,\bar{x})\, p(\bar{x},\bar{y}) z^k \right]$$

$$= \sum_{j=0}^{N} p(j\,;\,\bar{y}) z^j + \sum_{\nu=0}^{N} \sum_{\bar{x}\,\varepsilon\,F_\nu} z^{\nu-1} \left[P(z\,;\,\bar{x}) - \sum_{k=0}^{N+1-\nu} p(k\,;\,\bar{x}) z^k \right] p(\bar{x},\bar{y})$$

$$= \sum_{j=0}^{N} p(j ; \bar{y}) z^j + \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} P(z ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$- \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1}\, p(0 ; \bar{x})\, p(\bar{x}, \bar{y}) - \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} \sum_{k=1}^{N+1-v} p(k ; \bar{x})\, z^k\, p(\bar{x}, \bar{y})$$

$$= \sum_{j=0}^{N} p(j ; \bar{y}) z^j + \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} P(z ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$- \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1}\, p(0 ; \bar{x})\, p(\bar{x}, \bar{y}) - \sum_{k=1}^{N+1} \sum_{v=0}^{N+1-k} \sum_{\bar{x} \in F_v} z^{v-1} z^k\, p(k ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$= \sum_{j=0}^{N} p(j ; \bar{y}) z^j + \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} P(z ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$- \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1}\, p(0 ; \bar{x})\, p(\bar{x}, \bar{y}) - \sum_{k=1}^{N+1} \sum_{j=k-1}^{N} \sum_{\bar{x} \in F_{j-k+1}} z^j\, p(k ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$= \sum_{j=0}^{N} \sum_{k=1}^{j+1} \sum_{\bar{x} \in F_{j-k+1}} p(k ; \bar{x})\, z^j\, p(\bar{x}, \bar{y}) + \sum_{j=0}^{N} \sum_{\bar{x} \in F_j} p(0 ; \bar{x})\, z^j\, p(\bar{x}, \bar{y})$$

$$+ \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} P(z ; \bar{x})\, p(\bar{x}, \bar{y}) - \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1}\, p(0 ; \bar{x})\, p(\bar{x}, \bar{y})$$

$$- \sum_{j=0}^{N} \sum_{k=1}^{j+1} \sum_{\bar{x} \in F_{j-k+1}} p(k ; \bar{x})\, z^j\, p(\bar{x}, \bar{y})$$

So

$$P(z ; \bar{y}) = \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} \left[P(z ; \bar{x}) + (z-1)\, p(0 ; \bar{x}) \right] p(\bar{x}, \bar{y})$$

Appendix B

In this section we derive the additional equation that is shown in (13). We start by adding all the equations that are given by (9) and by using (8b) we obtain

$$P'(z) = \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} \left[(v-1)z^{v-2} [P(z ; \bar{x}) + (z-1)\, p(0 ; \bar{x})] + z^{v-1} [P'(z ; \bar{x}) + p(0 ; \bar{x})] \right]$$

By adding

$$z\, P'(z) - z \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} P'(z ; \bar{x}) = 0$$

in the first equation and rearranging we obtain

$$P'(z) = \frac{A(z)}{1-z} \tag{B1}$$

where

$$A(z) = \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} \left[(v-1)z^{v-2} [P(z ; \bar{x}) + (z-1)\, p(0 ; \bar{x})] + z^{v-1} [P'(z ; \bar{x}) + p(0 ; \bar{x})] - zP'(z ; \bar{x}) \right]$$

Since, $P'(1)$ is the average number of packets in the system, which is finite if (5) holds, and since $1-z = 0$ for $z=1$, we compute $P'(1)$ by applying L'Hospital's rule to (B1) ; thus

$$P'(z) \mid_{z=1} = \frac{dA(z)/dz}{d(1-z)/dz} \mid_{z=1} = - dA(z)/dz \mid_{z=1} \tag{B2}$$

From condition (B2) and by using (8_b) we obtain (13).

To calculate the steady state probability of having no packet in the system we proceed as before. We start by adding the equations given by (6) and by using (8_a) we obtain

$$P(z) = \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} z^{v-1} [P(z ; \bar{x}) + (z-1)\, p(0;\bar{x})]$$

By adding

$$zP(z) - z \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} P(z ; \bar{x}) = 0$$

in the previous equation and rearranging we obtain

$$P(z) = \frac{B(z)}{1-z} \tag{B3}$$

where

$$B(z) = \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} (z^{v-1} - z) [P(z ; \bar{x}) + (z-1)\, p(0 ; \bar{x})]$$

Since $P(1) = 1 < \infty$ and $1 - z = 0$ for $z = 1$, we compute P (1) by applying L'Hospital's rule to (B3); thus

$$1 = P(z) \mid_{z=1} = \frac{dB(z)/dz}{d(1-z)/dz} \mid_{z=1} = - dB(z)/dz \mid_{z=1} \tag{B4}$$

From condition (B4) and by using (7_b) we have

$$p_0 = 1 - \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} v\pi(\bar{x}) = 1 - \sum_{v=0}^{N} \sum_{\bar{x} \in F_v} v \prod_{i=1}^{N} \pi_i(x_i)$$

The previous expression after some manipulations results in (14).

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